The Fibonacci Quilt Game

Alexandra Newlon
Colgate University
anewlon@colgate.edu

Joint work with Neelima Borade, Catherine Wahlenmayer,
and Wanqiao Xu. Mentored by Steven J. Miller.

REU Conference
University of Massachusetts Amherst, July 23, 2019

Outline

The Fibonacci Sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55...$$

The Fibonacci Sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55...$$

Let $F_0 = F_1 = 1$, and for $n >= 2$

$$F_n = F_{n-1} + F_{n-2}$$

The Fibonacci Sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55...$$

Let $F_0 = F_1 = 1$, and for $n >= 2$

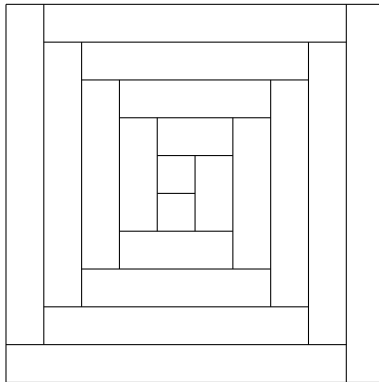$$F_n = F_{n-1} + F_{n-2}$$

Theorem (Zeckendorf)

Every positive integer can be written uniquely as the sum of non-consecutive Fibonacci numbers where
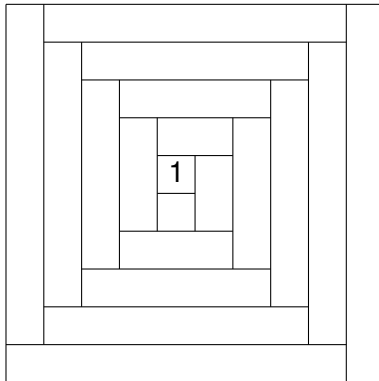
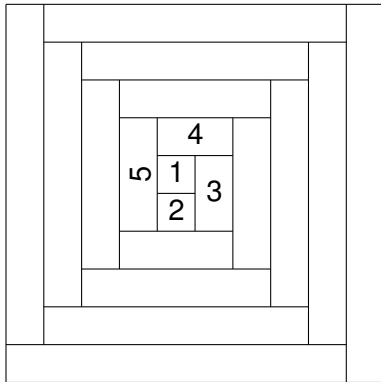$$F_n = F_{n-1} + F_{n-2}$$

and $F_1 = 1$, $F_2 = 2$.

The Fibonacci Quilt Sequence

The Fibonacci Quilt Sequence

The Fibonacci Quilt Sequence

The Fibonacci Quilt Sequence
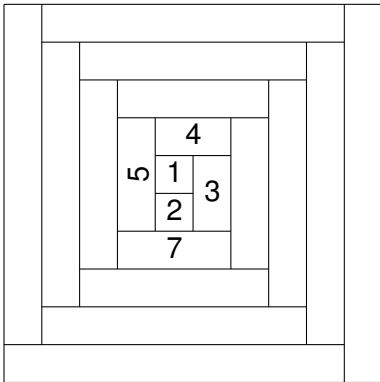
The Fibonacci Quilt Sequence
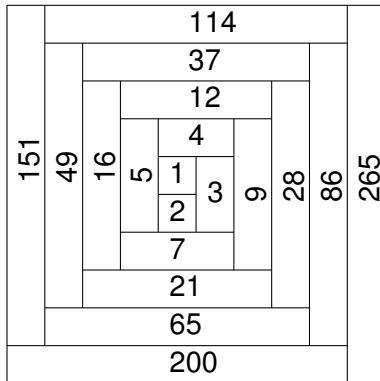
FQ-legal Decomposition

Definition (Catral, Ford, Harris, Miller, Nelson)

Let an increasing sequence of positive integers $q_i{}_{i=1}^{\infty}$ be given. We declare a decomposition of an integer

$$m = q_{l_1} + q_{l_2} + \cdots + q_{l_t}$$

(where $q_{l_i} > q_{l_{i+1}}$) to be an FQ-legal decomposition if for all i, j, $|l_i - l_j| \neq 0, 1, 3, 4$ and $\{1, 3\} \not\subset \{l_1, l_2, \ldots, l_t\}$.

The Fibonacci Quilt Sequence

Definition (Catral, Ford, Harris, Miller, Nelson)

The Fibonacci Quilt sequence is an increasing sequence of positive integers $\{q_i\}_{i=1}^{\infty}$, where every q_i ($i \geq 1$) is the smallest positive integer that does not have an FQ-legal decomposition using the elements $\{q_1, \ldots, q_{i-1}\}$.
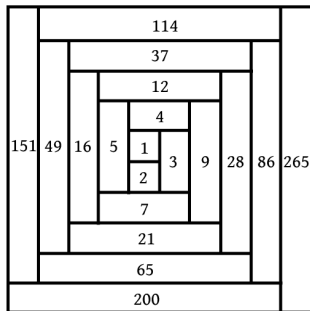
Recurrence Relations

Theorem (Catral, Ford, Harris, Miller, Nelson)

Let q_n denote the nth term in the Fibonacci Quilt, then

$$\text{for } n \geq 5, q_{n+1} = q_{n-1} + q_{n-2},$$

$$\text{for } n \geq 6, q_{n+1} = q_n + q_{n-4}.$$

General Rules

- Inspired by the Zeckendorf Game
- Two player game, alternate turns, last to move wins.
- Start with n 1's (q_1's)
- A turn is one of the following 4 general rules and some base rules.

General Rules

- Inspired by the Zeckendorf Game
- Two player game, alternate turns, last to move wins.
- Start with n 1's (q_1's)
- A turn is one of the following 4 general rules and some base rules.

Rule 1

$$\text{For } n \geq 2, q_n + q_{n+1} \to q_{n+3}$$
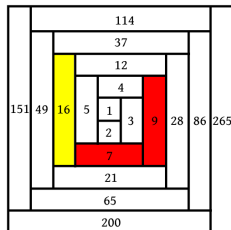
General Rules

Rule 2

$$\text{For } n \geq 2, q_n + q_{n+4} \rightarrow q_{n+5}$$
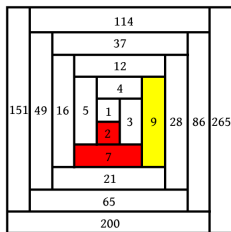
General Rules

Rule 3

$$2q_n \to q_{n+2} + q_{n-5} \text{ for } n \geq 7$$

For $n \geq 7,\ 2q_n \to q_{n+2} + q_{n-5}$
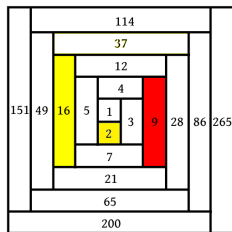
Genereal Rules

Rule 4

$$\text{For } n \geq 7, \ q_n + q_{n+3} \rightarrow q_{n-5} + q_{n+4}$$
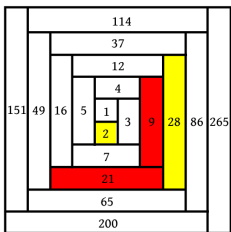
Base Rules

To handle base cases, we added additional base rules that

- preserves the total number of pieces
- does not produce violation of legality

Base Rules

To handle base cases, we added additional base rules that

- preserves the total number of pieces
- does not produce violation of legality

Special Rule

$$1 + 5 \rightarrow 2 + 4$$

Note: This rule can only be applied when nothing else can be done.

Example Game

1, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28...

n = 10 = 9 + 1

1	2	3	4	5	7	9
10	0	0	0	0	0	0

Example Game

1, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28...

n = 10 = 9 + 1

1	2	3	4	5	7	9
10	0	0	0	0	0	0
8	1	0	0	0	0	0

Example Game

1, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28...

n = 10 = 9 + 1

1	2	3	4	5	7	9
10	0	0	0	0	0	0
8	1	0	0	0	0	0
7	0	1	0	0	0	0

Example Game

1, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28...

n = 10 = 9 + 1

1	2	3	4	5	7	9
10	0	0	0	0	0	0
8	1	0	0	0	0	0
7	0	1	0	0	0	0
5	1	1	0	0	0	0

Example Game

1, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28...

$n = 10 = 9 + 1$

1	2	3	4	5	7	9
10	0	0	0	0	0	0
8	1	0	0	0	0	0
7	0	1	0	0	0	0
5	1	1	0	0	0	0
4	0	2	0	0	0	0

Example Game

1, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28...

$n = 10 = 9 + 1$

1	2	3	4	5	7	9
10	0	0	0	0	0	0
8	1	0	0	0	0	0
7	0	1	0	0	0	0
5	1	1	0	0	0	0
4	0	2	0	0	0	0
3	0	1	1	0	0	0

Example Game

1, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28...

n = 10 = 9 + 1

1	2	3	4	5	7	9
10	0	0	0	0	0	0
8	1	0	0	0	0	0
7	0	1	0	0	0	0
5	1	1	0	0	0	0
4	0	2	0	0	0	0
3	0	1	1	0	0	0
2	0	1	0	1	0	0

Example Game

1, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28...

$n = 10 = 9 + 1$

1	2	3	4	5	7	9
10	0	0	0	0	0	0
8	1	0	0	0	0	0
7	0	1	0	0	0	0
5	1	1	0	0	0	0
4	0	2	0	0	0	0
3	0	1	1	0	0	0
2	0	1	0	1	0	0
0	1	1	0	1	0	0

Example Game

1, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28...

n = 10 = 9 + 1

1	2	3	4	5	7	9
10	0	0	0	0	0	0
8	1	0	0	0	0	0
7	0	1	0	0	0	0
5	1	1	0	0	0	0
4	0	2	0	0	0	0
3	0	1	1	0	0	0
2	0	1	0	1	0	0
0	1	1	0	1	0	0
0	0	1	0	0	1	0

Example Game

1, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28...

$n = 10 = 9 + 1$

1	2	3	4	5	7	9
10	0	0	0	0	0	0
8	1	0	0	0	0	0
7	0	1	0	0	0	0
5	1	1	0	0	0	0
4	0	2	0	0	0	0
3	0	1	1	0	0	0
2	0	1	0	1	0	0
0	1	1	0	1	0	0
0	0	1	0	0	1	0
1	0	0	0	0	0	1

The Game is Playable

Theorem

Every game terminates in a finite number of moves at an FQ-legal decomposition.

The Game is Playable

Theorem

Every game terminates in a finite number of moves at an FQ-legal decomposition.

Proof Sketch: The sum of the squares of the indices of the terms is a strict monovarient.

The Game is Playable

Theorem

Every game terminates in a finite number of moves at an FQ-legal decomposition.

Proof Sketch: The sum of the squares of the indices of the terms is a strict monovarient.

- $q_n \wedge q_{n+1} \longrightarrow q_{n+3}$: $\sqrt{n+3} - \sqrt{n} - \sqrt{n+1} < 0$

The Game is Playable

Theorem

Every game terminates in a finite number of moves at an FQ-legal decomposition.

Proof Sketch: The sum of the squares of the indices of the terms is a strict monovarient.

- $q_n \wedge q_{n+1} \longrightarrow q_{n+3}$: $\sqrt{n+3} - \sqrt{n} - \sqrt{n+1} < 0$
- $q_n \wedge q_{n+4} \longrightarrow q_{n+5}$: $\sqrt{n+5} - \sqrt{n} - \sqrt{n+4} < 0$

The Game is Playable

Theorem

Every game terminates in a finite number of moves at an FQ-legal decomposition.

Proof Sketch: The sum of the squares of the indices of the terms is a strict monovarient.

- $q_n \wedge q_{n+1} \longrightarrow q_{n+3}$: $\sqrt{n+3} - \sqrt{n} - \sqrt{n+1} < 0$
- $q_n \wedge q_{n+4} \longrightarrow q_{n+5}$: $\sqrt{n+5} - \sqrt{n} - \sqrt{n+4} < 0$
- $2q_n \longrightarrow q_{n+2} \wedge q_{n-5}$: $\sqrt{n+2} + \sqrt{n-5} - 2\sqrt{n} < 0$

The Game is Playable

Theorem

Every game terminates in a finite number of moves at an FQ-legal decomposition.

Proof Sketch: The sum of the squares of the indices of the terms is a strict monovariant.

- $q_n \wedge q_{n+1} \longrightarrow q_{n+3}$: $\sqrt{n+3} - \sqrt{n} - \sqrt{n+1} < 0$
- $q_n \wedge q_{n+4} \longrightarrow q_{n+5}$: $\sqrt{n+5} - \sqrt{n} - \sqrt{n+4} < 0$
- $2q_n \longrightarrow q_{n+2} \wedge q_{n-5}$: $\sqrt{n+2} + \sqrt{n-5} - 2\sqrt{n} < 0$
- $q_n \wedge q_{n+3} \longrightarrow q_{n+4} \wedge q_{n-5}$:
  $\sqrt{n+4} + \sqrt{n-5} - \sqrt{n} - \sqrt{n+3} < 0$

The Last Move

1	2	3	4	5	7	9	12	16	21	28	37	49
1	0	0	0	1	0	?	0	0	?	?	?	?

The Last Move

1	2	3	4	5	7	9	12	16	21	28	37	49
1	0	0	0	1	0	?	0	0	?	?	?	?

If there is no 9:

1	2	3	4	5	7	9	12	16	21	28	37	49
1	0	0	0	1	0	0	0	0	?	?	?	?
0	1	0	1	0	0	0	0	0	?	?	?	?

The Last Move

1	2	3	4	5	7	9	12	16	21	28	37	49
1	0	0	0	1	0	?	0	0	?	?	?	?

If there is no 9:

1	2	3	4	5	7	9	12	16	21	28	37	49
1	0	0	0	1	0	0	0	0	?	?	?	?
0	1	0	1	0	0	0	0	0	?	?	?	?

If there is a 9:

1	2	3	4	5	7	9	12	16	21	28	37	49
1	0	0	0	1	0	1	0	0	0	0	?	?
0	1	0	1	0	0	1	0	0	0	0	?	?
1	1	0	0	0	0	0	1	0	0	0	?	?

Lower Bound on Game Length

Notation

Let $L(n)$ denote the maximum number of terms in an FQ-legal decomposition of n. Let $l(n)$ denote the minimum number of terms in an FQ-legal deocomposition of n.

Examples:
$20 = 16 + 4 = 12 + 7 + 1$
$L(20) = 3,\ l(20) = 2$

Lower Bound on Game Length

Notation

Let $L(n)$ denote the maximum number of terms in an FQ-legal decomposition of n. Let $l(n)$ denote the minimum number of terms in an FQ-legal deocomposition of n.

Examples:
$20 = 16 + 4 = 12 + 7 + 1$
$L(20) = 3, l(20) = 2$
$50 = 49 + 1 = 28 + 16 + 4 + 2$
$L(50) = 4, l(50) = 2$

Lower Bound on Game Length

Theorem

The shortest possible game on n is completed in $n - L(n)$ moves.

Lower Bound on Game Length

Theorem

The shortest possible game on n is completed in $n - L(n)$ moves.

Proof Sketch: Induction on n.

Lower Bound on Game Length

Theorem

The shortest possible game on n is completed in $n - L(n)$ moves.

Proof Sketch: Induction on n.
If n is in the Fibonacci Quilt Sequence, denoted q_i

$$q_{i-3} + q_{i-2} = q_i$$

Lower Bound on Game Length

Theorem

The shortest possible game on n is completed in $n - L(n)$ moves.

Proof Sketch: Induction on n.

If n is in the Fibonacci Quilt Sequence, denoted q_i

$$q_{i-3} + q_{i-2} = q_i$$

If n is not in the sequence

$$n = q_{i_1} + q_{i_2} + \cdots + q_{i_{L(n)}}$$

Number of moves:

$$(q_{i_1} - 1) + (q_{i_2} - 1) + \cdots + (q_{i_{L(n)}} - 1)$$
$$= (q_{i_1} + q_{i_2} + \cdots + q_{i_{L(n)}}) - L(n)$$
$$= n - L(n)$$

Distribution of Game Lengths

Conjecture

The distribution of a random game length approaches Gaussian as n increases.
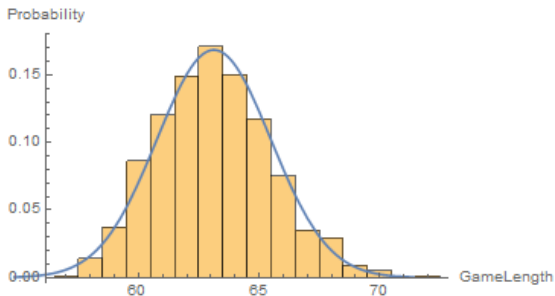

Figure: Distribution of 1000 games on n=60

Future Work

- Is there a deterministic game that always results in the lower bound?

- What patterns emerge
  in the winner of certain deterministic games as n increases?

- Does either player have a winning strategy?
  - Analogous result on the Zeckendorf Game shows that for $n > 2$, player 2 has a winning strategy

Thank You

References

- M. Catral, P.L. Ford, P.E. Harris, S.J. Miller, D. Nelson, *Legal Decomposition Arising From Non-Positive Linear Recurrences*. Fibonacci Quarterly (54 (2016), no. 4, 348-365).
- P. Baird-Smith, A. Epstein, K. Flint, S.J. Miller, *The Zeckendorf Game*. (2018).

Thank you to Dr. Miller (NSF Grant DMS1561945), the SMALL program (NSF Grant DMS1659037) and Williams College.